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DYNAMIC STABILITY OF THE FLEXURAL VIBRATIONS OF A THIN-WALLED BEAM

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Abstract—The stability of the free transverse vibrations of a simply supported, thin-walled elastic beam with double axes of symmetry is investigated for arbitrary initial velocity distributions. The lateral-torsional modes may be parametrically excited through the interaction of the pulsating flexural moment and the rotation of the cross section of the beam about the longitudinal axis and the vertical axis of symmetry. A criterion is presented to predict this excitation. A numerical example for a particular transverse flexural mode of vibration indicates that the critical initial velocity for lateral-torsional parametric excitation may be relatively small.

1. INTRODUCTION

A CHARACTERISTIC of thin-walled beams of open section is their weak torsional resistance in comparison with beams of closed profile. The dynamic stability of the transverse vibrations of these unbraced members is an important practical consideration due to the severe lateral-torsion oscillations that may ensue as result of nonlinear coupling. Initially the predominate nonlinear coupling is the interaction of the pulsating transverse flexural moment and the rotations about the longitudinal axis and the vertical axis of symmetry.

Bolotin [1] investigated the dynamic stability of the plane flexural modes of vibrations due to simple harmonic loadings. The stability of the steady-state vibrations is governed by the stability of the Mathieu equation.

Here we consider the dynamic stability of the transverse vibrations due to an impulsive loading of arbitrary spatial distribution. The effect of such a loading is to impart an initial velocity to the beam, but other initial conditions could be included by introducing an appropriate phase angle. Due to the arbitrariness of the spatial distribution of the impulsive loading, all the flexural modes of vibration will be excited. Therefore, vibrations containing harmonics of all the natural frequencies must be considered, whereas only oscillations of a single frequency occur for simple harmonic loadings.

Although arbitrary boundary conditions can be treated, only the simply supported beam will be considered. In addition, coincidence of the shear center and centroid of the section is assumed. The latter assumption can be relaxed and the method extended to beams with a single longitudinal plane of symmetry and with the unperturbed motion in this plane.

Since both lateral and torsional modes of oscillation can exhibit growth, this investigation represents an extension of previous analyses of dynamic stability of impulsively loaded structures (e.g. see [2, 3, 4]).

2. DERIVATION OF GOVERNING EQUATIONS

We consider a simply supported, thin-walled, elastic beam of open section and of length L. The origin of the xyz reference is positioned at the left end of the beam with the

z-axis directed along the centroidal axis. The cross section possesses double axes of symmetry. The displacement of the beam is described by the displacements u and v of the centroid of the section and the rotation β as indicated in Fig. 1. These quantities are functions of the coordinate z and time t.

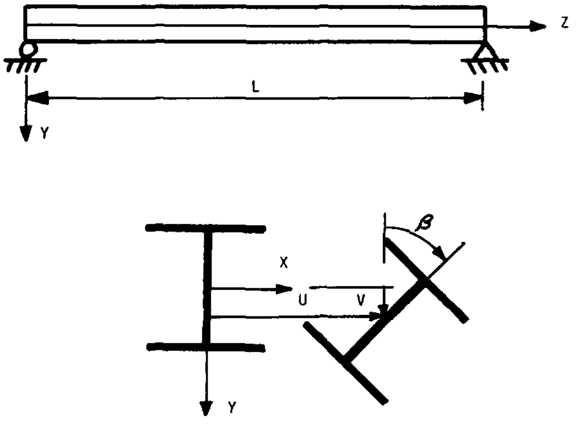

FIG. 1. Geometry of beam.

The displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ of any point in the beam of rigid cross section [5] are

$$\bar{u} = u - y\beta, \qquad \bar{v} = v + x\beta$$
$$\bar{w} = -xu_{,z} - yv_{,z} + \phi\beta_{,z} \tag{2.1}$$

where ϕ is the warping function. For the envisioned motion $\bar{w}_{,x}$, $\bar{w}_{,y}$, $\bar{w}_{,z} \ll 1$. The non-vanishing components of strain are

$$\begin{aligned} \varepsilon_{zz} &= \bar{w}_{,z} + \tfrac{1}{2}(\bar{u}_{,z}^2 + \bar{v}_{,z}^2) \\ \varepsilon_{xz} &= \tfrac{1}{2}(\bar{u}_{,z} + \bar{w}_{,x}) + \tfrac{1}{2}(\bar{u}_{,x}\bar{u}_{,z} + \bar{v}_{,x}\bar{v}_{,z}) \\ \varepsilon_{yz} &= \tfrac{1}{2}(\bar{v}_{,z} + \bar{w}_{,y}) + \tfrac{1}{2}(\bar{u}_{,y}\bar{u}_{,z} + \bar{v}_{,y}\bar{v}_{,z}). \end{aligned} \tag{2.2}$$

We determine the equations of motion according to Hamilton's principle. The potential energy is

$$V = \frac{1}{2}\int_0^L \int_A [E\varepsilon_{zz}^2 + 4G(\varepsilon_{xy}^2 + \varepsilon_{yz}^2)]\, \mathrm{d}A\, \mathrm{d}z. \tag{2.3}$$

Introduction of (2.1) and (2.2) into (2.3) leads to

$$\begin{aligned} V = \frac{1}{2}\int_0^L & [EI_x(v_{,zz} + u_{,z}\beta_{,z})^2 + EI_y(u_{,zz} - v_{,z}\beta_{,z})^2 \\ & + E\Gamma\beta_{,zz}^2 + GK\beta_{,z}^2 + \frac{EA}{4}(u_{,z}^2 + v_{,z}^2 + r^2\beta_{,z}^2)^2 \\ & + GA\beta^2(u_{,z}^2 + v_{,z}^2 + r^2\beta_{,z}^2) + EB\beta_{,zz}\beta_{,z}^2 + \frac{E}{4}(C - r^4A)\beta_{,z}^4]\, \mathrm{d}z. \end{aligned} \tag{2.4}$$

In (2.4) EI_x and EI_y, $E\Gamma$, and GK are the flexural, warping, and torsional rigidities respectively. The geometric quantities r^2, B, C are defined as

$$r^2 = (I_x+I_y)/A, \qquad B = \int_A (x^2+y^2)\phi \,\mathrm{d}A, \qquad C = \int_A (x^2+y^2)^2 \,\mathrm{d}A. \tag{2.5}$$

For typical beams the rotary inertia may be discarded [5], whereupon the expression for the kinetic energy simplifies to

$$T = \frac{\rho A}{2}\int_0^L (u_{,t}^2+v_{,t}^2+r^2\beta_{,t}^2)\,\mathrm{d}z. \tag{2.6}$$

The following expressions satisfy the simply supported boundary conditions

$$\begin{aligned} u &= L\sum_{n=1}^{\infty} x_n(\tau)\sin n\pi s \\ v &= L\sum_{n=1}^{\infty} y_n(\tau)\sin n\pi s \\ \beta &= \sum_{n=1}^{\infty} \beta_n(\tau)\sin n\pi s \end{aligned} \tag{2.7}$$

in which

$$\tau = ct/L, \qquad c = (E/\rho)^{\frac{1}{2}}, \qquad s = z/L. \tag{2.8}$$

Substituting (2.7) into (2.4) and (2.6) we obtain from Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T-V)}{\partial q_i} = 0$$

the governing differential equations for x_n, y_n, β_n by taking each in turn as a generalized coordinate q_i

$$\begin{aligned} &\ddot{y}_n+\omega_{ny}^2 y_n+f_{ny}(y_i, x_j, \beta_k) = 0 \\ &\ddot{x}_n+\omega_{nx}^2 x_n+n\pi^3\alpha_x\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} k^2\bar{a}_{kl}^n y_k\beta_l+f_{nx}(y_i, x_j, \beta_k) = 0 \\ &\ddot{\beta}_n+\omega_{n\beta}^2 \beta_n+n\pi^3\alpha_x\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} k^2\bar{b}_{kl}^n y_k x_l+f_{n\beta}(y_i, x_j, \beta_k) = 0, \end{aligned} \tag{2.9}$$

$$n = 1, 2, \ldots$$

where

$$\begin{aligned} \bar{a}_{kl}^n &= \left[\alpha_x\frac{l(k+l)}{(k+l)^2-n^2}+\frac{\alpha_y^2}{\alpha_x}\frac{l}{k}\frac{n^2}{(k+l)^2-n^2}\right. \\ &\quad \left.+\alpha_x\frac{l(k-l)}{(k-l)^2-n^2}+\frac{\alpha_y^2}{\alpha_x}\frac{l}{k}\frac{n^2}{(k-l)^2-n^2}\right][(-1)^{k+l+n}-1] \\ \bar{b}_{kl}^n &= \left[\frac{\alpha_x}{\alpha^2}\frac{l(k+l)}{(k+l)^2-n^2}-\frac{\alpha_y^2}{\alpha^2\alpha_x}\frac{l^2}{k}\frac{(k+l)}{(k+l)^2-n^2}\right. \\ &\quad \left.+\frac{\alpha_x}{\alpha^2}\frac{l(k-l)}{(k-l)^2-n^2}+\frac{\alpha_y^2}{\alpha^2\alpha_x}\frac{l^2}{k}\frac{(k-l)}{(k-l)^2-n^2}\right][(-1)^{k+l+n}-1]. \end{aligned} \tag{2.10}$$

Whenever $(k+l)^2 = n^2$, the first two terms within the first set of brackets of (2.10) are replaced by zero and the last two terms are replaced by zero for $(k-l)^2 = n^2$. The dot indicates ordinary differentiation with respect to τ and f_{ny}, f_{nx}, $f_{n\beta}$ are higher order polynomials in x_i, y_j, β_k. With the neglect of the longitudinal inertia no nonlinear terms involving y_j alone will occur in f_{ny} if one end is free to move axially. In addition, we have introduced the following dimensionless quantities:

$$\alpha_x^2 = I_x/AL^2, \qquad \alpha_y^2 = I_y/AL^2, \qquad \alpha^2 = (r/L)^2 \tag{2.11}$$

$$\omega_{nx} = n^2\pi^2\alpha_y, \qquad \omega_{ny} = n^2\pi^2\alpha_x,$$
$$\omega_{n\beta} = \frac{n\pi}{r}\left[\frac{n^2\pi^2\Gamma}{AL^2} + \frac{K}{2(1+v)A}\right]^{\frac{1}{2}} \tag{2.12}$$

v being Poisson's ratio.

The unperturbed transverse motion is produced by an initial velocity whose Fourier expansion is

$$\frac{\partial v(s,0)}{\partial t} = \sum_{n=1}^{\infty} a_n \sin n\pi s. \tag{2.13}$$

This motion is further perturbed by small lateral and rotational velocities

$$\frac{\partial u(s,0)}{\partial t} = \sum_{n=1}^{\infty} \alpha_n a_n \sin n\pi s$$
$$\frac{\partial \beta(s,0)}{\partial t} = L \sum_{n=1}^{\infty} \gamma_n a_n \sin n\pi s \tag{2.14}$$

where $\alpha_n, \gamma_n \ll 1$. From (2.7) and (2.14) it is evident that x_n, y_n, β_n must satisfy

$$y_n(0) = 0 \qquad \dot{y}_n(0) = \frac{a_n}{c}$$
$$x_n(0) = 0 \qquad \dot{x}_n(0) = \frac{\alpha_n a_n}{c} \tag{2.15}$$
$$\beta_n(0) = 0 \qquad \dot{\beta}_n(0) = \frac{\gamma_n a_n}{c}.$$

3. PARAMETRIC EXCITATION

Initially the x_n's and β_n's are small since their initial disturbance is assumed small. Therefore, the higher order polynomials in (2.9) may be discarded in the initial phase of the motion. The solution of the first of (2.9) which satisfies (2.15) is

$$y_n = \frac{a_n}{n^2\pi^2\alpha_x c}\sin\omega_{ny}\tau, \qquad n = 1, 2, \ldots. \tag{3.1}$$

Substituting (3.1) into (2.9) and defining an initial velocity parameter as

$$\varepsilon = \frac{\pi a_1}{c} \tag{3.2}$$

we obtain

$$
\begin{aligned}
\ddot{x}_n + \omega_{nx}^2 x_n &= -n\varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^n \beta_l \sin \omega_{ky}\tau, \qquad n = 1, 2, \ldots \\
\ddot{\beta}_m + \omega_{m\beta}^2 \beta_m &= -m\varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl}^m x_l \sin \omega_{ky}\tau, \qquad m = 1, 2, \ldots
\end{aligned}
\tag{3.3}
$$

in which

$$
\begin{Bmatrix} a_{kl}^n \\ b_{kl}^n \end{Bmatrix} = \frac{a_k}{a_1} \begin{Bmatrix} \bar{a}_{kl}^n \\ \bar{b}_{kl}^n \end{Bmatrix}. \tag{3.4}
$$

The appearance of the initial transverse velocity as a parameter in the perturbed equations (3.3) indicates that the lateral and torsional modes may be parametrically excited. These equations differ from the typical equations governing parametric excitation of other impulsively loaded structural elements [2, 3, 4]. The right-hand side of (3.3) is proportional to the nth (mth) component of the bending (twisting) moment which is equal to the product of the transverse pulsating bending moment and the rotation about the longitudinal axis (rotation about the vertical axis) of the beam.

The stability or instability of the transverse oscillations is governed by the stability or instability of the trivial solution of (3.3) $x_n = \beta_m = 0$. One notable method of determining the stability criterion for system of differential equations with periodic coefficients is due to Hsu [6, 7]. It is convenient to write (3.3) as

$$
\begin{aligned}
z_n &= \dot{x}_n \\
\dot{z}_n + \omega_{nx}^2 x_n &= -n\varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^n \sin \omega_{ky}\tau \beta_l \\
\theta_m &= \dot{\beta}_m \\
\dot{\theta}_m + \omega_{m\beta}^2 \beta_m &= -m\varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl}^m \sin \omega_{ky}\tau x_l.
\end{aligned}
\tag{3.5}
$$

For elastic behavior the parameter $\varepsilon \ll 1$. The solution of (3.5) is assumed as the sum of a variation of parameters solution and a perturbation solution in ε; i.e.

$$
\begin{aligned}
x_n &= A_n(\tau)\cos \omega_{nx}\tau + B_n(\tau)\sin \omega_{nx}\tau + \varepsilon f_n(\tau) + \sum_{j=2}^{N} \varepsilon^j f_{nj}(\tau) \\
z_n &= -\omega_{nx}A_n(\tau)\sin \omega_{nx}\tau + \omega_{nx}B_n(\tau)\cos \omega_{nx}\tau + \varepsilon \dot{f}_n(\tau) + \sum_{j=2}^{N} \varepsilon^j \dot{f}_{nj}(\tau) \\
\beta_m &= C_m(\tau)\cos \omega_{m\beta}\tau + D_m(\tau)\sin \omega_{m\beta}\tau + \varepsilon g_m(\tau) + \sum_{j=2}^{N} \varepsilon^j g_{mj}(\tau) \\
\theta_m &= -\omega_{m\beta}C_m(\tau)\sin \omega_{m\beta}\tau + \omega_{m\beta}D_m(\tau)\cos \omega_{m\beta}\tau + \varepsilon \dot{g}_m(\tau) + \sum_{j=2}^{N} \varepsilon^j \dot{g}_{mj}(\tau).
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) into (3.5) and retaining terms of first order in ε, we have for the first approximation

$$\begin{aligned}
&\dot{A}_n \cos \omega_{nx}\tau + \dot{B}_n \sin \omega_{nx}\tau = 0 \\
&-\omega_{nx}\dot{A}_n \sin \omega_{nx}\tau + \omega_{nx}\dot{B}_n \cos \omega_{nx}\tau + \varepsilon(\ddot{f}_n + \omega_{nx}^2 f_n) \\
&= -\frac{n\varepsilon}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^n \{C_l[\sin(\omega_{ky}+\omega_{l\beta})\tau + \sin(\omega_{ky}-\omega_{l\beta})\tau] \\
&+ D_l[\cos(\omega_{ky}-\omega_{l\beta})\tau - \cos(\omega_{ky}+\omega_{l\beta})\tau]\}, \qquad (n = 1, 2, \ldots) \\
&\dot{C}_m \cos \omega_{m\beta}\tau + \dot{D}_m \sin \omega_{m\beta}\tau = 0 \\
&-\omega_{m\beta}\dot{C}_m \sin \omega_{m\beta}\tau + \omega_{m\beta}\dot{D}_m \cos \omega_{m\beta}\tau + \varepsilon(\ddot{g}_m + \omega_{m\beta}^2 g_m) \\
&= -\frac{m\varepsilon}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl}^m \{A_l[\sin(\omega_{ky}+\omega_{lx})\tau + \sin(\omega_{ky}-\omega_{lx})\tau] \\
&+ B_l[\cos(\omega_{ky}-\omega_{lx})\tau - \cos(\omega_{ky}+\omega_{lx})\tau]\}, \qquad (m = 1, 2, \ldots).
\end{aligned} \tag{3.7}$$

Significant deviation from the unperturbed motion can occur whenever the system is near "resonance". The resonant conditions are

$$\omega_{nx} = \omega_{ky}+\omega_{l\beta}, \quad \omega_{nx} = |\omega_{ky}-\omega_{l\beta}| \tag{3.8a,b}$$

$$\omega_{m\beta} = \omega_{ky}+\omega_{lx}, \quad \omega_{m\beta} = |\omega_{ky}-\omega_{lx}| \tag{3.9a,b}$$

where we require $(k \pm l)^2 \neq n^2$ and $(k \pm l)^2 \neq m^2$ in (3.8) and (3.9) respectively. Equation (3.8) expresses the condition of resonance between a harmonic of the lateral mode and the lateral component of the unperturbed bending moment. A similar interpretation exists for (3.9).

For a given n we assume there exists one pair of integers i, j that most nearly satisfies (3.8a) and a second pair p, q that most nearly satisfies (3.8b). We write

$$\begin{aligned}
\omega_{nx} &= \omega_{iy}+\omega_{j\beta}+\Delta_1 \\
\omega_{nx} &= |\omega_{py}-\omega_{q\beta}|+\Delta_2
\end{aligned} \tag{3.10}$$

where Δ_1 and Δ_2 are considered small compared to ω_{nx} and are a measure of the detuning; i.e. the deviation of the actual frequencies from the resonant conditions. It is obvious that (3.10) reduces to (3.8) when Δ_1 and Δ_2 vanish.

Returning to (3.7) we associate the resonant terms of the right-hand side with the variational part of the solution and obtain accordingly

$$\begin{aligned}
&\dot{A}_n \cos \omega_{nx}\tau + \dot{B}_n \sin \omega_{nx}\tau = 0 \\
&-\omega_{nx}\dot{A}_n \sin \omega_{nx}\tau + \omega_{nx}\dot{B}_n \cos \omega_{nx}\tau \\
&= -\frac{n\varepsilon}{2}\{a_{ij}^n[C_j \sin(\omega_{nx}-\Delta_1)\tau - D_j \cos(\omega_{nx}-\Delta_1)\tau] \\
&+ a_{pq}^n[\pm C_q \sin(\omega_{nx}-\Delta_2)\tau + D_q \cos(\omega_{nx}-\Delta_2)\tau]\}
\end{aligned}$$

$$\begin{aligned}
&\dot{C}_j \cos \omega_{j\beta}\tau + \dot{D}_j \sin \omega_{j\beta}\tau = 0 \\
&-\omega_{j\beta}\dot{C}_j \sin \omega_{j\beta}\tau + \omega_{j\beta}\dot{D}_j \cos \omega_{j\beta}\tau \\
&= -\frac{j\varepsilon b^j_{in}}{2}[-A_n \sin(\omega_{j\beta}+\Delta_1)\tau + B_n \cos(\omega_{j\beta}+\Delta_1)\tau] \\
&\dot{C}_q \cos \omega_{q\beta}\tau + \dot{D}_q \sin \omega_{q\beta}\tau = 0 \\
&-\omega_{q\beta}\dot{C}_q \sin \omega_{q\beta}\tau + \omega_{q\beta}\dot{D}_q \cos \omega_{q\beta}\tau \\
&= -\frac{q\varepsilon b^q_{pn}}{2}[A_n \sin(\omega_{q\beta} \mp \Delta_2)\tau \pm B_n \cos(\omega_{q\beta} \pm \Delta_2)\tau]
\end{aligned} \tag{3.11}$$

where the upper sign is to be used if $\omega_{py} > \omega_{q\beta}$ and the lower sign when $\omega_{py} < \omega_{q\beta}$. Equations (3.11) can be simplified by the Kryloff–Bogoliuboff–Van der Pol method to a form which readily lends itself to integration. Solving for the first derivatives and averaging over their respective periods, we obtain the complex representation

$$\begin{aligned}
\dot{X}_n &= \frac{n\varepsilon}{4\omega_{nx}}[a^n_{ij} Y_j \, e^{i\Delta_1\tau} \pm a^n_{pq} Z_q \, e^{i\Delta_2\tau}] \\
\dot{Y}_j &= -\frac{j\varepsilon b^j_{in}}{4\omega_{j\beta}} X_n \, e^{-i\Delta_1\tau} \\
\dot{Z}_q &= \frac{q\varepsilon b^q_{pn}}{4\omega_{q\beta}} X_n \, e^{-i\Delta_2\tau}
\end{aligned} \tag{3.12}$$

where

$$X_n = A_n + iB_n, \qquad Y_j = C_j + iD_j, \qquad Z_q = C_q \pm iD_q. \tag{3.13}$$

No confusion should arise between the imaginary number $i = \sqrt{-1}$ and the integer i since the latter occurs only as a subscript. Solutions to (3.12) are taken in the form

$$X_n = X_{on} \, e^{i\lambda\tau}, \qquad Y_j = Y_{oj} \, e^{i(\lambda-\Delta_1)\tau}, \qquad Z_q = Z_{oq} \, e^{i(\lambda-\Delta_2)\tau}. \tag{3.14}$$

The characteristic equation for λ is

$$\lambda^3 - (\Delta_1+\Delta_2)\lambda^2 + \left(\frac{\pm nq\varepsilon^2 a^n_{pq} b^q_{pn}}{16\omega_{nx}\omega_{q\beta}} - \frac{jn\varepsilon^2 a^n_{ij} b^j_{in}}{16\omega_{nx}\omega_{j\beta}} + \Delta_1\Delta_2\right)\lambda + \frac{jn\varepsilon^2 a^n_{ij} b^j_{in}\Delta_2}{16\omega_{nx}\omega_{j\beta}} \mp \frac{nq\varepsilon^2 a^n_{pq} b^q_{pn}\Delta_1}{16\omega_{nx}\omega_{q\beta}} = 0. \tag{3.15}$$

It is apparent from the form of (3.14) that the condition that perturbations do not grow, and hence for the transverse motion to be stable, is that the roots of (3.15) have positive imaginary parts. Since the complex roots of (3.15) always occur in conjugate pairs, then the motion is stable if and only if the roots are real and distinct. Setting

$$\begin{aligned}
P &= -(\Delta_1+\Delta_2) \\
Q &= \pm\frac{nq\varepsilon^2 a^n_{pq} b^q_{pn}}{16\omega_{nx}\omega_{q\beta}} - \frac{jn\varepsilon^2 a^n_{ij} b^j_{in}}{16\omega_{nx}\omega_{j\beta}} + \Delta_1\Delta_2 \\
R &= \frac{jn\varepsilon^2 a^n_{ij} b^j_{in}\Delta_2}{16\omega_{nx}\omega_{j\beta}} \mp \frac{nq\varepsilon^2 a^n_{pq} b^q_{pn}\Delta_1}{16\omega_{nx}\omega_{q\beta}}
\end{aligned} \tag{3.16}$$

the well-known condition that (3.15) have real distinct roots is

$$\frac{b^2}{4}+\frac{a^3}{27} < 0 \tag{3.17}$$

where

$$\begin{aligned} a &= \tfrac{1}{3}(3Q-P^2) \\ b &= \tfrac{1}{27}(2P^3-9PQ+27R). \end{aligned} \tag{3.18}$$

A necessary condition for stability is $P^2 > 3Q$ or

$$\frac{3n\varepsilon^2}{16\omega_{nx}}\left[\pm\frac{qa^n_{pq}b^q_{pn}}{\omega_{q\beta}}-\frac{ja^n_{ij}b^j_{in}}{\omega_{j\beta}}\right] < \Delta_1^2+\Delta_2^2-\Delta_1\Delta_2. \tag{3.19}$$

Introducing (3.18) into (3.17) we have

$$Q^3-\tfrac{1}{4}Q^2P^2+\tfrac{27}{4}R^2-\tfrac{9}{2}PQR+RP^3 < 0. \tag{3.20}$$

For some values of n the resonant condition (3.8a) is not physically realizable. In this case we set $a^n_{ij} = b^j_{in} = \Delta_1 = 0$ and (3.20) reduces to

$$Q-\tfrac{1}{4}P^2 < 0. \tag{3.21}$$

Due to the symmetry of (3.3) and the resonant conditions (3.8) and (3.9) the stability criterion associated with (3.9) is obtained by replacing x, a^n_{ij}, b^j_{in}, n and β by β, b^n_{ij}, a^j_{in}, m and x, respectively, in the foregoing.

For a given beam and initial velocity distribution the critical initial velocity is obtained from (3.20) or (3.21) by considering successive values of n. If the sum of i, j, n is an even integer, then this case is unimportant since a^n_{ij} and b^j_{in} vanish. There may be other values of i and j which satisfy equally well the same resonant conditions for the same n. In principle this can be treated in a manner similar to the foregoing, but finding the solution to the variational equations is further complicated by the fact that their number increases by two for every pair of i and j. If the convergence of the Fourier expansion of the initial velocity is sufficiently rapid, it will be the set of integers with the smallest values that will be significant. Similar discussion applies to p and q.

4. EXAMPLE

As a numerical example we consider a beam of rectangular section whose length to depth ratio is 20, depth to width ratio is 10, and Poisson's ratio is 1/3. We will limit ourselves to the resonant condition (3.8) and $n = 3$. We find that (3.8a) is not physically realizable and from (3.8b) we obtain $p = q = 3$ and $\Delta_2 = 4{\cdot}05\times10^{-4}$ with $\omega_{py} > \omega_{q\beta}$. The small value of Δ_2 compared to ω_{3x} indicates we are very near resonance. From (3.21) we find the critical value of ε for $n = 3$ to be

$$\varepsilon_{cr} = 7{\cdot}85\times10^{-5}\frac{a_1}{a_3} \quad \text{or equivalently} \quad (a_3)_{cr} = 2{\cdot}5\times10^{-5}c.$$

We note that a very small initial velocity will cause the transverse motion to become unstable.

5. FUTURE WORK

The parametric excitation of the lateral and torsional modes will undoubtedly be more important in beams possessing only a single axis of symmetry. In this case there will be additional twisting and bending moments due to noncoincidence of the shear center and the centroid of the section. In this case direct coupling between the transverse vibrations and the lateral and torsional modes of vibration will exist in addition to the indirect coupling witnessed here.

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Абстракт—Исследуется устойчивость свободных, поперечных колебаний свободно опертой, тонкостенной, упругой балки с двумя осями симметрии для определения распределений произвольной начальной скорости. Поперечно-крутильные формы колебаний могут быть параметрически вызваны путем взаимодействия между пульсирующим моментом изгиба и вращением поперечного сечения балки, вокрут продольной и вертикальной осей симметрии. Дается критерий для определения этого возбуждения. Числовой пример для частной поперечно-изгибной формы колебаний указывает, что критическое начальное, поперечно-крутильное, параметрическое возбуждение может быть относительно малым.